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A class of coupled KdV systems and their bi-Hamiltonian formulation

Wen–Xiu Ma†

Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, People's Republic of China

Abstract. A Hamiltonian pair is proposed and thus a type of hereditary operators results. All the corresponding systems of evolution equations possess local bi-Hamiltonian formulation and a special choice of the systems leads to the KdV hierarchy. Illustrative examples are given.

Bi-Hamiltonian formulation is significant for investigating integrable properties of nonlinear systems of differential equations [1–3]. Many mathematical and physical systems have been found to possess such bi-Hamiltonian formulation. There are two important problems related to bi-Hamiltonian theory. First, which kind of systems can possess bi-Hamiltonian formulation and secondly, how to construct bi-Hamiltonian formulation for a given system if it exists. There are no complete answers to these two problems as yet, although a lot of general analysis for bi-Hamiltonian formulation itself has been made. However, we should make as many observations on structures of various bi-Hamiltonian systems as possible, so that these matters may finally be resolved.

Therefore, in order to enhance our understanding of bi-Hamiltonian formulation, we would like to search for new examples of bi-Hamiltonian systems among coupled KdV systems and their higher-order partners. There are already some theories which allow us to do that. For instance, we can generate soliton hierarchies by using decomposable hereditary operators [4] or by using perturbation around solutions [5]. In this paper, we simply wish to present some new concrete examples to satisfy the Magri scheme [1] by considering decomposable hereditary operators.

Let us choose two specific matrix differential operators:

$$J=\begin{bmatrix} 0 & & & \alpha_0\partial \\ & & \alpha_0\partial & \alpha_1\partial \\ & \iddots & \iddots & \vdots \\ \alpha_0\partial & \alpha_1\partial & \cdots & \alpha_N\partial \end{bmatrix} \qquad M=\begin{bmatrix} 0 & & & M_0 \\ & & M_0 & M_1 \\ & \iddots & \iddots & \vdots \\ M_0 & M_1 & \cdots & M_N \end{bmatrix} \tag{1}$$

with

$$\partial=\frac{\partial}{\partial x} \qquad M_i=c_i\partial^3+d_i\partial+2u_{ix}+4u_i\partial \qquad u_i=u_i(x,t) \qquad 0\leqslant i\leqslant N \tag{2}$$

where α_i, c_i, d_i, $0\leqslant i\leqslant N$, are arbitrary constants, but $\alpha_0\neq 0$ which guarantees the invertibility of J. It is known [6] that J and M constitute a pair of Hamiltonian operators with respect to the potential vector $u=(u_0,u_1,\ldots,u_N)^T$, that is to say, $aJ+bM$ is a

† E-mail address: mawx@cityu.edu.hk

Hamiltonian operator for any two constants a and b, which may also be proved directly by the Gel'fand–Dorfman algebraic method [2, 7].

Now we can generate a hereditary operator $\Phi = MJ^{-1}$ (see, say, [4] for a general proof), since J is invertible. To express this operator explicitly, we need to compute the inverse of J. In view of the specific form of J, we can assume

$$J^{-1} = \begin{bmatrix} \beta_0\partial^{-1} & \beta_1\partial^{-1} & \cdots & \beta_N\partial^{-1} \\ \beta_1\partial^{-1} & & \iddots & \\ \vdots & \iddots & & \\ \beta_N\partial^{-1} & & & 0 \end{bmatrix} \tag{3}$$

where β_i, $0 \leqslant i \leqslant N$, are constants to be determined. It simply follows that

$$JJ^{-1} = (J^{-1}J)^T = \begin{bmatrix} \alpha_0\beta_N & & & 0 \\ \alpha_0\beta_{N-1} + \alpha_1\beta_N & \alpha_0\beta_N & & \\ \vdots & \ddots & \ddots & \\ \alpha_0\beta_0 + \alpha_1\beta_1 + \cdots + \alpha_N\beta_N & \cdots & \alpha_0\beta_{N-1} + \alpha_1\beta_N & \alpha_0\beta \end{bmatrix}.$$

Therefore $JJ^{-1} = J^{-1}J = I_{N+1}$, where I_{N+1} is an identity matrix operator of size $(N+1) \times (N+1)$, leads to an equivalent system of linear algebraic equations for β_i, $0 \leqslant i \leqslant N$:

$$\alpha_0\beta_N = 1, \alpha_0\beta_{N-1} + \alpha_1\beta_N = 0, \ldots, \alpha_0\beta_0 + \alpha_1\beta_1 + \cdots + \alpha_N\beta_N = 0 \tag{4}$$

which may be written as

$$A\beta = E_1 \qquad \text{i.e.} \begin{bmatrix} 0 & & & \alpha_0 \\ & & \alpha_0 & \alpha_1 \\ & \iddots & \iddots & \vdots \\ \alpha_0 & \alpha_1 & \cdots & \alpha_N \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5}$$

The coefficient matrix A is invertible since $\alpha_0 \neq 0$, and thus this linear system has a unique solution $\beta = A^{-1}E_1$. Now we can obtain

$$\Phi = MJ^{-1} = \begin{bmatrix} \beta_N\Phi_0 & & & 0 \\ \beta_{N-1}\Phi_0 + \beta_N\Phi_1 & \beta_N\Phi_0 & & \\ \vdots & \ddots & \ddots & \\ \beta_0\Phi_0 + \beta_1\Phi_1 + \cdots + \beta_N\Phi_N & \cdots & \beta_{N-1}\Phi_0 + \beta_N\Phi_1 & \beta_N\Phi_0 \end{bmatrix} \tag{6}$$

where

$$\Phi_i = M_i\partial^{-1} = c_i\partial^2 + d_i + 2u_{ix}\partial^{-1} + 4u_i \qquad 0 \leqslant i \leqslant N \tag{7}$$

and then the conjugate operator of Φ reads as

$$\Psi = \Phi^\dagger = \begin{bmatrix} \beta_N\Psi_0 & \beta_{N-1}\Psi_0 + \beta_N\Psi_1 & \cdots & \beta_0\Psi_0 + \beta_1\Psi_1 + \cdots + \beta_N\Psi_N \\ & \ddots & \ddots & \vdots \\ & & \beta_N\Psi_0 & \beta_{N-1}\Psi_0 + \beta_N\Psi_1 \\ 0 & & & \beta_N\Psi_0 \end{bmatrix} \tag{8}$$

where

$$\Psi_i = \Phi_i^\dagger = c_i\partial^2 + d_i + 2u_i + 2\partial^{-1}u_i\partial \qquad 0 \leqslant i \leqslant N. \tag{9}$$

Because the Lie derivative of Φ with respect to u_x is zero, i.e.

$$L_{u_x}\Phi = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \Phi(u + \varepsilon u_x) - [I_{N+1}\partial, \Phi] = 0$$

we have (see, say, [1, 8–10])

$$[K_m, K_n] = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (K_m(u+\varepsilon K_n) - K_n(u+\varepsilon K_m)) = 0$$
$$K_n := \Phi^n u_x = 0 \qquad m, n \geqslant 0. \tag{10}$$

This also implies that a hierarchy of systems of evolution equations $u_t = K_n$, $n \geqslant 0$, has infinitely many common commuting symmetries $\{K_m\}_0^\infty$. All systems in the hierarchy have a common recursion operator Φ, since the operator Φ is hereditary and has a zero Lie derivative with respect to u_x: $L_{u_x}\Phi = 0$. Moreover, due to the specific forms of Φ_i, $0 \leqslant i \leqslant N$, they are all local, although the recursion operator Φ is integro-differential. A mathematical induction process may easily verify this statement on locality.

In what follows, we want to show local bi-Hamiltonian formulation for all systems except the first one in the hierarchy (note that sometimes systems of soliton equations have only one local Hamiltonian structure in bi-Hamiltonian formulation: such examples are the modified KdV equations and O(3) chiral field equations [11]).

First of all, we observe the second system

$$u_t = K_1 = \Phi u_x = M J^{-1} u_x.$$

The vector field $J^{-1}u_x$ can be computed as follows

$$\begin{aligned}
J^{-1}u_x &= \begin{bmatrix} \beta_0\partial^{-1} & \beta_2\partial^{-1} & \cdots & \beta_N\partial^{-1} \\ \beta_1\partial^{-1} & & \iddots & \\ \vdots & \iddots & & \\ \beta_N\partial^{-1} & & & 0 \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{1x} \\ \vdots \\ u_{Nx} \end{bmatrix} \\
&= \begin{bmatrix} \beta_0 u_0 + \beta_1 u_1 + \cdots + \beta_N u_N \\ \beta_1 u_0 + \beta_2 u_1 + \cdots + \beta_N u_{N-1} \\ \vdots \\ \beta_{N-1}u_0 + \beta_N u_1 \\ \beta_N u_0 \end{bmatrix} \\
&= \beta_N \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} + \beta_{N-1} \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \\ 0 \end{bmatrix} + \cdots + \beta_0 \begin{bmatrix} u_0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
&\stackrel{\text{def}}{=} \beta_N X_0 + \beta_{N-1} X_1 + \cdots + \beta_0 X_N.
\end{aligned} \tag{11}$$

Evidently we can find or directly prove that

$$f_0 := J^{-1}u_x = \frac{\delta \tilde{H}_0}{\delta u} \tag{12}$$

$$\tilde{H}_0 = \int H_0 \, \mathrm{d}x \qquad H_0 = \int_0^1 \langle f_0(\lambda u), u\rangle \, \mathrm{d}\lambda = \tfrac{1}{2}\sum_{l=0}^{N} \beta_l \sum_{i+j=l} u_i u_j \tag{13}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{N+1}$. This means that f_0 is gradient. Now we check whether or not the vector field Ψf_0 is a gradient field, which is required in the Magri scheme [1]. A direct computation can give

$$f_{1m} := \Psi X_m = \frac{\delta \tilde{H}_{1m}}{\delta u} \qquad \tilde{H}_{1m} = \int_{-\infty}^{\infty} H_{1m} \, \mathrm{d}x$$

$$H_{1m} = \int_0^1 \langle f_{1m}(\lambda u), u\rangle \, \mathrm{d}\lambda$$
$$= \sum_{l=m}^{N} \beta_l \sum_{i+j+k=l-m} [\tfrac{1}{2}(c_i u_j u_{kxx} + d_i u_j u_k) + u_i u_j u_k] \qquad 0 \leqslant m \leqslant N.$$

These equalities yield

$$f_1 := \Psi f_0 = \frac{\delta \tilde{H}_1}{\delta u} \tag{14}$$

where the Hamiltonian functional $\tilde{H}_1$ is determined by

$$\tilde{H}_1 = \int H_1 \, \mathrm{d}x \qquad H_1 = \sum_{m=0}^{N} \beta_{N-m} \sum_{l=m}^{N} \beta_l \sum_{i+j+k=l-m} [\tfrac{1}{2}(c_i u_j u_{kxx} + d_i u_j u_k) + u_i u_j u_k]. \tag{15}$$

Therefore the system $u_t = K_1 = \Phi u_x$ has local bi-Hamiltonian formulation

$$u_t = K_1 = \Phi u_x = J \frac{\delta \tilde{H}_1}{\delta u} = M \frac{\delta \tilde{H}_0}{\delta u} \tag{16}$$

where $\tilde{H}_0$ and $\tilde{H}_1$ are defined by (13) and (15), respectively.

Secondly, we want to expose bi-Hamiltonian formulation for the other systems $u_t = K_n$, $n \geqslant 2$. Note that $\Phi = \Psi^\dagger$ is hereditary, and that f_0 and Ψf_0 are already gradient. According to the Magri scheme [1, 4], all vector fields $\Psi^n f_0$, $n \geqslant 0$, are gradient fields, in other words there exists a hierarchy of functionals $\tilde{H}_n$, $n \geqslant 0$, such that

$$f_n := \Psi^n f_0 = \frac{\delta \tilde{H}_n}{\delta u} \qquad n \geqslant 0. \tag{17}$$

In fact, the Hamiltonian functionals $\tilde{H}_n$, $n \geqslant 0$, must be equal to

$$\tilde{H}_n = \int H_n \, \mathrm{d}x \qquad H_n = \int_0^1 \langle f_n(\lambda u), u\rangle \, \mathrm{d}\lambda \qquad n \geqslant 0 \tag{18}$$

and they are all in involution with respect to either Poisson bracket:

$$\{\tilde{H}_m, \tilde{H}_n\}_J := \int \frac{\delta \tilde{H}_m}{\delta u} J \frac{\delta \tilde{H}_n}{\delta u} \, \mathrm{d}x = 0 \qquad m, n \geqslant 0 \tag{19}$$

$$\{\tilde{H}_m, \tilde{H}_n\}_M := \int \frac{\delta \tilde{H}_m}{\delta u} M \frac{\delta \tilde{H}_n}{\delta u} \, \mathrm{d}x = 0 \qquad m, n \geqslant 0. \tag{20}$$

Thus all vector fields K_n, $n \geqslant 1$, can be written in two ways as

$$K_n = \Phi^n u_x = \Phi^n J f_0 = J \Psi^n f_0 = J \frac{\delta \tilde{H}_n}{\delta u} \qquad n \geqslant 1$$
$$K_n = \Phi^n u_x = (J\Psi)\Psi^{n-1} f_0 = M \Psi^{n-1} f_0 = M \frac{\delta \tilde{H}_{n-1}}{\delta u} \qquad n \geqslant 1$$

which provide local bi-Hamiltonian formulation

$$u_t = K_n = \Phi^n u_x = J \frac{\delta \tilde{H}_n}{\delta u} = M \frac{\delta \tilde{H}_{n-1}}{\delta u} \qquad n \geqslant 1 \tag{21}$$

for all systems $u_t = K_n$, $n \geqslant 1$. It follows that the systems $u_t = K_n$, $n \geqslant 0$, have infinitely many common commuting symmetries $\{K_m\}_0^\infty$ and conserved densities $\{H_m\}_0^\infty$, which justifies that they constitute a typical soliton hierarchy.

Note that the coefficients appearing in our construction are all arbitrary except the requirement of $\alpha_0 \neq 0$. Thus the resulting systems may contain many interesting systems. A special choice of

$$\alpha_0 = c_0 = 1 \qquad \alpha_i = c_i = 0 \qquad 1 \leqslant i \leqslant N \qquad d_i = 0 \qquad 0 \leqslant i \leqslant N$$

leads to the KdV hierarchy under the reduction $u_i = 0$, $1 \leqslant i \leqslant N$, and thus the above resulting systems are called coupled KdV systems.

Let us now show some examples. Let $N = 0$ and $\alpha_0 = 1$. We now have

$$J = \partial \qquad M = c_0\partial^3 + d_0\partial + 2u_{0x} + 4u_0\partial \qquad \Phi = c_0\partial^2 + d_0 + 2u_{0x}\partial^{-1} + 4u_0.$$

When $c_0 \neq 0$, $d_0 = 0$, the corresponding hierarchy is the KdV hierarchy. When $c_0 = d_0 = 0$, the corresponding hierarchy is a hierarchy of quasi-linear partial differential equations, of which the first two nonlinear equations are

$$u_t = \Phi u_x = 6u_0u_{0x} \qquad u_t = \Phi^2 u_x = 30u_0^2u_{0x}.$$

All the vector fields and all the Hamiltonian functionals in this hierarchy are of special form $cu_0^m u_{0x}$ and cu_0^m, where c is a constant and $m \in \mathbb{N}$, respectively.

Let $N = 1$. The corresponding Hamiltonian pair and hereditary operator become

$$J = \begin{bmatrix} 0 & \alpha_0\partial \\ \alpha_0\partial & \alpha_1\partial \end{bmatrix} \qquad M = \begin{bmatrix} 0 & c_0\partial^3 + d_0\partial + u_{0x} + 4u_0\partial \\ c_0\partial^3 + d_0\partial + u_{0x} + 4u_0\partial & c_1\partial^3 + d_1\partial + u_{1x} + 4u_1\partial \end{bmatrix}$$

$$\Phi = MJ^{-1} = M\begin{bmatrix} -\frac{\alpha_1}{\alpha_0^2}\partial^{-1} & \frac{1}{\alpha_0}\partial^{-1} \\ \frac{1}{\alpha_0}\partial^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_0}\Phi_0 & 0 \\ -\frac{\alpha_1}{\alpha_0^2}\Phi_0 + \frac{1}{\alpha_0}\Phi_1 & \frac{1}{\alpha_0}\Phi_0 \end{bmatrix}$$

where $\Phi_0 = c_0\partial^2 + d_0 + 2u_{0x}\partial^{-1} + 4u_0$ and $\Phi_1 = c_1\partial^2 + d_1 + 2u_{1x}\partial^{-1} + 4u_1$. The first nonlinear system is the following

$$u_t = \Phi u_x = J\frac{\delta\tilde{H}_1}{\delta u} = M\frac{\delta\tilde{H}_0}{\delta u}$$
$$= \begin{bmatrix} \frac{1}{\alpha_0}(c_0u_{0xxx} + d_0u_{0x} + 6u_0u_{0x}) \\ -\frac{\alpha_1}{\alpha_0^2}(c_0u_{0xxx} + d_0u_{0x} + 6u_0u_{0x}) + \frac{1}{\alpha_0}[(c_0u_1 + c_1u_0)_{xxx} + (d_0u_1 + d_1u_0)_x + 6(u_0u_1)_x] \end{bmatrix}$$

where the Hamiltonian functionals read as

$$\tilde{H}_0 = \int H_0\,\mathrm{d}x \qquad H_0 = \frac{1}{\alpha_0}u_0u_1 - \frac{\alpha_1}{2\alpha_0^2}u_0^2$$

$$\tilde{H}_1 = \int H_1\,\mathrm{d}x \qquad H_1 = \frac{1}{\alpha_0^2}\left(\frac{c_1}{2} - \frac{c_0\alpha_1}{\alpha_0}\right)u_0u_{0xx} + \frac{1}{\alpha_0^2}\left(\frac{d_1}{2} - \frac{d_0\alpha_1}{\alpha_0}\right)u_0^2 - \frac{2\alpha_1}{\alpha_0^3}u_0^3$$
$$+\frac{1}{\alpha_0^2}\left[\frac{c_0}{2}(u_0u_{1xx} + u_{0xx}u_1) + d_0u_0u_1 + 3u_0^2u_1\right].$$

For a general case of N, if we choose

$$\alpha_0 = 1 \qquad \alpha_i = 0 \qquad 1 \leqslant i \leqslant N$$
$$c_0 = 1 \qquad c_i = 0 \qquad 0 \leqslant i \leqslant N$$
$$d_i = 0 \qquad 0 \leqslant i \leqslant N$$

then the resulting systems are exactly the perturbation systems of the KdV hierarchy introduced in [12] through perturbation around solutions of the KdV equation.

It should be realized that all first nonlinear systems ($u_t = K_1 = \Phi u_x$) belong to a more general class of integrable coupled KdV systems, which was introduced by Gürses and Karasu in [13], motivated by the Jordan KdV systems in [14]. Moreover the principle parts of our coupled KdV systems, i.e. the systems with $d_i = 0$, $0 \leqslant i \leqslant N$, belong to a symmetric subclass in the non-degenerate case in [13]. This may be seen by observing the coefficients

$$b^i_j = \sum_{k=0}^{N} \beta_{N-i+j+k} c_k \qquad a^i_{jk} = 2c^i_{jk} = \frac{2}{3} s^i_{jk} = 4\beta_{N-i+j+k} \qquad 0 \leqslant i, j, k \leqslant N \tag{22}$$

where $\beta_i = 0$, $i < 0$ or $i > N$, are accepted, after our recursion operators and our coupled systems in the case of $d_i = 0$, $0 \leqslant i \leqslant N$, are rewritten as follows

$$\Phi(u) = (R^i_j)_{(N+1)\times(N+1)} \qquad R^i_j = b^i_j \partial^2 + \sum_{k=0}^{N} (a^i_{jk} u_k + c^i_{jk} u_{kx} \partial^{-1}) \tag{23}$$

$$u_{it} = \Phi(u) u_x = \sum_{k=0}^{N} b^i_k u_{kxxx} + \sum_{k,j=0}^{N} s^i_{jk} u_j u_{kx} \qquad 0 \leqslant i \leqslant N. \tag{24}$$

Actually the coefficients defined by (22) satisfy the relations

$$\sum_{k=0}^{N} b^k_l s^i_{jk} = \sum_{k=0}^{N} b^i_k s^k_{jl} \qquad \sum_{k=0}^{N} s^i_{jk} s^k_{lm} = \sum_{k=0}^{N} s^i_{lk} s^k_{jm} \qquad 0 \leqslant i, j, l, m \leqslant N$$

which guarantees [13] that the operators defined by (23) with the coefficients $a^i_{jk} = 2c^i_{jk} = \frac{2}{3} s^i_{jk}$ are recursion operators for the systems determined by (24) in the symmetric case of $s^i_{jk} = s^i_{kj}$.

The other nonlinear systems in each hierarchy determined by a hereditary operator Φ may contain much higher-order derivatives of u with respect to x, but they still have a recursion operator and even bi-Hamiltonian formulation. By taking a scaling transformation $t \to at$, $x \to bx$, $u \to cu$, more concrete examples of integrable coupled KdV systems [13] can be obtained from our systems.

Compared with the well known coupled KdV systems (for example, see [15–17]), the above systems are not really coupled because of the first separated component. The Lax pairs or the spectral problems associated with our systems have not been found yet. If they are found, master symmetries of the systems can also be presented like those of the well known coupled KdV systems in [17].

Using an idea of extension in [18], we can obtain much more general Hamiltonian pairs starting from the one above. There are also other choices of basic Hamiltonian operators, such as

$$\begin{bmatrix} r_x + 2r\partial & s\partial \\ s_x + s\partial & 0 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} \partial + \begin{bmatrix} 0 & c_4 \\ -c_4 & 0 \end{bmatrix} \partial^2$$

$$u = \begin{bmatrix} r \\ s \end{bmatrix} \qquad c_i = \text{constants} \qquad 1 \leqslant i \leqslant 4$$

in [19], which lead to new Hamiltonian pairs and then new integrable systems having bi-Hamiltonian formulation. However, we need more techniques in manipulation.

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